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A THEORY OF VISCOPLASTICITY

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Abstract—An alternative to the currently used Perzyna's theory of viscoplasticity for small strains is presented. The plastic strain rate vector is normal to the quasistatic yield surface which in turn may not include the origin. Consequently, new loading/unloading criteria are introduced.

1. INTRODUCTION

THE purpose of the present paper is to discuss a theory of viscoplasticity in the small strain range which is to a considerable extent different from the theory presented by Perzyna [1]. The concept of the viscoplastic material has been introduced and developed by several writers, a few of which are Bingham [2], Hohenemser and Prager [3], Hencky [4], Oldroyd [5], Fromm [6], Naghdi and Murch [7] and most recently Perzyna [1].§ It is not the purpose of this paper to give a survey of the historical development of the theory of viscoplasticity. Consequently we shall not present the individual contributions of the above mentioned and other authors to the present state of the theory. Since, however Perzyna's version of the theory of viscoplasticity seems to be the most well-known one we shall give a short account of it in order to emphasize in what respects the theory developed in this paper differs from that of Perzyna's. The comparison between the present theory and Perzyna's infinitesimal theory is appropriate since the latter has been widely used for the solution of many boundary value problems.

Perzyna's theory is based on the existence of a yield function

$$F(\sigma_{ij}, \varepsilon_{ij}^{vp}, k) = \frac{f(\sigma_{ij}, \varepsilon_{ij}^{vp})}{k} - 1 \tag{1}$$

where the function $f(\sigma_{ij}, \varepsilon_{ij}^{vp})$ depends on the stress σ_{ij} and on the viscoplastic strain ε_{ij}^{vp}. The parameter k is defined by the expression

$$k = k(W_{vp}) = k\left(\int_0^{\varepsilon_{ij}^{vp}} \sigma_{ij}\, d\varepsilon_{ij}^{vp}\right) \tag{2}$$

and it is the strain hardening parameter. The yield condition is given by

$$F = 0 \tag{3}$$

or

$$f(\sigma_{ij}, \varepsilon_{ij}^{vp}) = k \tag{4}$$

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§ Recent papers by Perzyna and Wojno [8], Mandel [9], Kratochvill and Dillon [10] and Valanis [11] are for finite strain.

so that the viscoplastic strain rate $\dot{\varepsilon}_{ij}^{vp}$ is zero when $F \leq 0$ (or $f \leq k$) and finite when $F > 0$ (or $f > k$).

The viscoplastic strain rate is given by

$$\dot{\varepsilon}_{ij}^{vp} = \gamma^0 \langle \Phi(F) \rangle \frac{\partial F}{\partial \sigma_{ij}} \tag{5}$$

where γ^0 is a material constant and

$$\langle \Phi(F) \rangle = \begin{cases} 0 & \text{for } F \leq 0 \\ \Phi(F) & \text{for } F > 0. \end{cases} \tag{6}$$

The function $\Phi(F)$ is selected such that it will agree with experimental results.

Equation (5) may also be written as

$$\dot{\varepsilon}_{ij}^{vp} = \gamma \langle \Phi(F) \rangle \frac{\partial f}{\partial \sigma_{ij}} \tag{7}$$

where $\gamma = \gamma^0/k$, which now is a function of the viscoplastic work since it depends on k.

Some remarks are now in order. The function $f = k$ or $F = 0$ is the yield surface and on this surface the viscoplastic strain rate is zero. At the stress point outside the yield surface, $\dot{\varepsilon}_{ij}^{vp}$ is different than zero and it is normal to the surface $f = c^*$ where c^* is the value f takes when the coordinates of the stress point are inserted into f. The surface $f = c^*$ as well as the surface $f = k$ are two members of the family of surfaces $f = c$ where f is a given function of the stress and viscoplastic strain and c is a scalar parameter. The surfaces $f = c$ are in general not parallel to one another, so that $\dot{\varepsilon}_{ij}^{vp}$ is normal only to the surface $f = c^*$, passing through the stress point, which we may call the *dynamic loading surface.* The surface $f = k$ could be denoted the *quasistatic yield surface* since it separates the region of stress space where viscoplastic strains can appear from the region where viscoplastic strains are zero. These two surfaces are related since they are obtained from each other by a change in the numerical value of the scalar parameter c. It should be emphasized that $c = k$ is a very definite value dealing with the viscoplastic work done up to the time considered. On the other hand the other values of c, including c^* the one through which the dynamic loading surface is created, have no physical meaning.

Figure 1 shows the two surfaces. We remark that when the normal to the dynamic loading intersects the quasistatic yield surface we may write

$$\sigma_{ij} = \lambda_{ij} + 2\eta \dot{\varepsilon}_{ij}^{vp} \tag{8}$$

where λ_{ij} is the stress at the intersection of the normal at σ_{ij} to $f = c^*$ with the surface $f = k$. Here η is a parameter which can be determined by using equation (7). We obtain

$$\sigma_{ij} = \lambda_{ij} + 2\eta\gamma \langle \Phi(F) \rangle \frac{\partial f}{\partial \sigma_{ij}} \tag{9}$$

For equations (8) and (9) to be possible it is necessary that the normal to the surface $f = c^*$ at σ_{ij} intersects the surface $f = k$. This, of course, is not always possible as it is seen from Fig. 1.

The theory presented in this paper is to some extent a generalization of ideas presented earlier by Fromm [6] and by Prager [12]. Our theory assumes the existence of a quasistatic yield surface and it assumes that the viscoplastic strain rate is a function of the excess

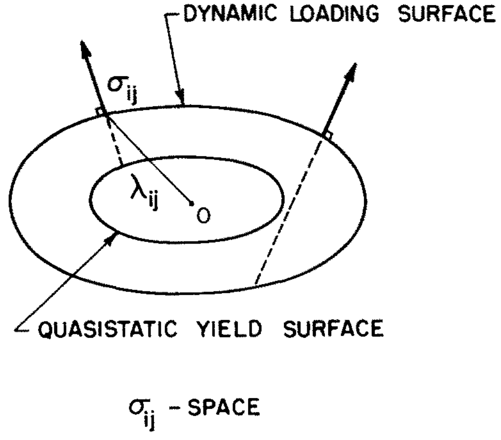

FIG. 1. The quasistatic yield surface and the dynamic loading surface (Perzyna's theory).

stress to be defined in the next section. The viscoplastic strain rate is not normal to a dynamic loading surface but it is normal to the quasistatic yield surface. The latter assumption is, we believe, reasonable since a quasistatic yield surface is a well defined surface in plasticity while the concept of a dynamic loading surface is still to some extent uncertain. For this reason, the dynamic loading surface plays only a minor role in the development of our theory. In this paper we shall consider the general case that the quasistatic yield surface may or may not enclose the origin. New loading/unloading criteria must therefore be introduced.

The difference between the present theory and Perzyna's theory is two-fold. Not only the constitutive equations but also the dynamic loading surfaces are different. In Perzyna's theory, the current dynamic loading surface is always an isotropic expansion of the corresponding subsequent static yield surface. This is not the case in the present theory.

In the remainder of this paper we shall develop the theory to some degree of completeness. Application of this theory to boundary value problems is described in two papers in preparation. It is shown that the proposed theory is very simple to apply. Finally in a fourth paper in preparation this theory is compared to available experimental results and needed experimentation is proposed.

2. THE FUNDAMENTAL EQUATIONS

In this paper we shall consider a rigid–viscoplastic material under the condition of small strain.† We postulate the existence of a quasistatic yield surface $F = 0$ such that the material is rigid when the stress point lies within or on the surface $F = 0$, and it is viscoplastic when the stress point lies outside the surface $F = 0$.

It is assumed that the hydrostatic pressure does not cause yielding and does not influence the viscoplastic properties of the material. We also assume that no viscoplastic work is done by the hydrostatic pressure; hence there is no viscoplastic change in volume. Finally it is

† The elastic strains can be accounted for without any conceptual difficulties. The loading/unloading criterion must, however, be slightly modified if the elastic strains must be accounted for.

assumed that the influence of the strain rate history can be neglected. The term *excess stress* denoted by h, is defined as the perpendicular distance in the deviatoric stress space from the current stress point A to the quasistatic yield surface, Fig. 2. Let s_{ij} be the stress deviator given by

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk}. \tag{10}$$

The stress deviator representing the foot of the normal from the stress point A to the quasistatic yield surface is denoted by k_{ij}. The *excess stress* tensor h_{ij} is given by

$$s_{ij} - k_{ij} = h_{ij} = hn_{ij} \tag{11}$$

where n_{ij} is the unit normal vector to the quasistatic yield surface $F = 0$ at k_{ij} so that

$$n_{ij} = \frac{\partial F/\partial k_{ij}}{[(\partial F/\partial k_{lm})(\partial F/\partial k_{lm})]^{1/2}}. \tag{12}$$

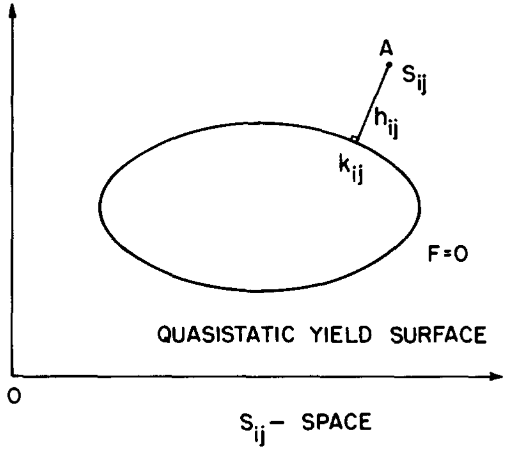

FIG. 2. The quasistatic yield surface (New theory).

It is assumed that h is a function of k_{rs}, of the viscoplastic deviatoric strain e_{rs}^{vp} and of the second invariant of the viscoplastic deviatoric strain rate I_2^{vp}, and the viscoplastic strain rate $\dot{e}_{ij}^{vp}$ due to a quasistatic or to a dynamic loading is normal to the quasistatic yield surface. Hence we shall write

$$\begin{aligned} \dot{e}_{ij}^{vp} &= \gamma(k_{rs}, e_{rs}^{vp}, I_2^{vp})\Phi[h(k_{rs}, e_{s}^{vp}, I_2^{vp})]n_{ij}, \quad &\text{for } h > 0 \\ \dot{e}_{ij}^{vp} &= 0 &\text{for } h = 0 \end{aligned} \tag{13}$$

where $\gamma(k_{rs}, e_{rs}^{vp}, I_2^{vp})$ is a viscosity function which is dependent on the position k_{rs}, the viscoplastic strain e_{rs}^{vp} and I_2^{vp}. The function Φ is a scalar function of h.

For a certain amount of excess stress h the strain rate is determined by (13). On the other hand, if the strain rate is prescribed, the excess stress is also determined. Thus, the distance between the loading point and the quasistatic yield surface is determined by the strain rate.

Equation (11) can now be written as

$$s_{ij} = k_{ij}(e_{rs}) + h(k_{rs}, e_{rs}, I_2^{vp})n_{ij}. \tag{14}$$†

† For rigid–viscoplastic materials, the total strain e_{rs} is same as the viscoplastic strain e_{rs}^{vp}.

Expression (14) defines the dynamic loading surface at each value of e_{rs} (i.e. at each degree of strain hardening) and at each value of I_2^{vp} that is at each value of the strain rate. Any surface which encloses the quasistatic yield surface and has a distance h from it, h being a function of k_{rs}, e_{rs} and I_2^{vp}, is a dynamic loading surface. A convenient definition of a dynamic loading surface is that on the dynamic loading surface the second invariant of the strain-rate tensor is constant. With this definition the dynamic loading surface will reduce to the quasistatic yield surface when the strain-rate is very low. Normality between the plastic strain rate due to quasistatic loading and the quasistatic yield surface is assumed as is usually done in plasticity. Since h is a function of position, the normal to the dynamic loading surface and the strain-rate vector $\dot{e}_{ij}^{vp}$ are not in the same direction. It is here assumed that the direction of $\dot{e}_{ij}$ due to dynamic loading is normal to the quasistatic yield surface (coincident with the direction of $\dot{e}_{ij}$ due to quasistatic loading).

Multiplying both sides of equation (13) by itself and taking the square root of the resulting expression we obtain

$$(I_2)^{1/2} = (1/\sqrt{(2)})\gamma\Phi(h). \tag{15}$$

From (15) we see that $\gamma\Phi(h)$ has to remain constant throughout the entire dynamic loading surface.

Equation (15) can be inverted again to obtain

$$h = \Phi^{-1}\left[\frac{\sqrt{2}}{\gamma}(I_2)^{1/2}\right] \tag{16}$$

where Φ^{-1} denotes the inverse function of Φ. Expression (16) provides us with a scalar relationship between I_2 and h for a given $\gamma(k_{rs}, e_{rs}, I_2^{vp})$, which can be determined by experiments. Rewriting equation (15) in the form

$$\gamma = \frac{\sqrt{2}}{\Phi(h)}(I_2)^{1/2} \tag{17}$$

we see that the viscosity function γ is dependent on the position, strain and strain rate.

3. THE CHANGES IN THE DYNAMIC LOADING SURFACE DUE TO QUASISTATIC AND DYNAMIC PRELOADING

We shall now simplify the theory and assume that the viscosity function γ is a constant and that the excess stress h is independent of position k_{ij} and viscoplastic strain e_{ij}^{vp}. Hence h is a function of I_2^{vp} only. Then the dynamic loading surfaces are parallel to the quasistatic yield surface and the strain rate vector is normal to both surfaces. Thus the previous equations are simplified to

$$s_{ij} = k_{ij}(e_{rs}) + hn_{ij} \tag{18}$$

$$\begin{aligned} \dot{e}_{ij} &= \gamma\Phi(h)n_{ij}, \qquad \text{for } h > 0 \\ \dot{e}_{ij} &= 0, \qquad \text{for } h = 0 \end{aligned} \tag{19}$$

$$h = \Phi^{-1}\left[\frac{\sqrt{2}}{\gamma}(I_2)^{1/2}\right]. \tag{20}$$

We observe that for the viscoplastic material the viscoplastic strain needs time to appear, it will not recover once it has appeared and it can also be obtained by quasistatic straining. Hence, as the first approximation it is assumed that the subsequent quasistatic yield surface, and the subsequent dynamic material behavior, are not affected by the prestrain-rate, but are affected by the plastic strain produced. This assumption is equivalent to the basic assumption of the present theory that the strain rate history has negligible effects. Thus, for a given plastic strain, any strain-hardening theory of plasticity may be employed to obtain the subsequent static yield surface. The subsequent dynamic loading surfaces can then be constructed based upon this newly obtained subsequent static yield surface. In this manner, the infinitesimal change of the static deviatoric stress dk_{ij} due to ds_{ij} can easily be obtained, and similarly we obtain the quantity ds_{ij} due to dk_{ij}.

For clarity, let us consider the one-dimensional stress-strain curve of Fig. 3. Let a dynamic loading of strain rate $\dot{e}_1$ follow the curve $OECA$. A sudden removal of the stress takes place at A which follow the curve $AA'D$ until the point D is reached. No additional plastic strain has been produced due to this sudden drop of stress. A reloading of the same strain rate $\dot{e}_1$ will follow $DA'A$ back to point A and further loading of the same strain rate will then follow the curve AG, which is a continuation of the original curve CA. A further loading of strain rate other than $\dot{e}_1$ will, however, not follow AG.

If an infinitesimal dynamic load $\mathrm{d}s$ is applied at A, an infinitesimal increase of plastic strain $\mathrm{d}e = \dot{e}\,\mathrm{d}t$ is produced in time interval $\mathrm{d}t$ and the stress point will move from A to B. Points A' and B' are two points on the static curve which measure respectively the degree of strain hardening of the stress states A and B. The change of the static yield stress $\mathrm{d}k$ due to $\mathrm{d}s$ is shown geometrically in Fig. 3. Physically, $\mathrm{d}k$ is an infinitesimal stress which when applied statically at A' will produce the same amount of plastic strain $\mathrm{d}e$ as that produced by $\mathrm{d}s$ above.

The idea illustrated by Fig. 3 can be extended to describe the application of the infinitesimal dynamic loading $\mathrm{d}s_{ij}$ in the multi-dimensional case, and this is shown in Fig. 4 in a multi-dimensional deviatoric stress space.

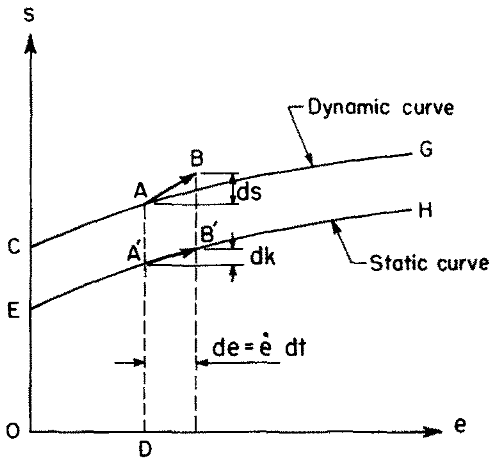

FIG. 3. The plastic strain produced by dynamic loading obtained by straining the material quasi-statically.

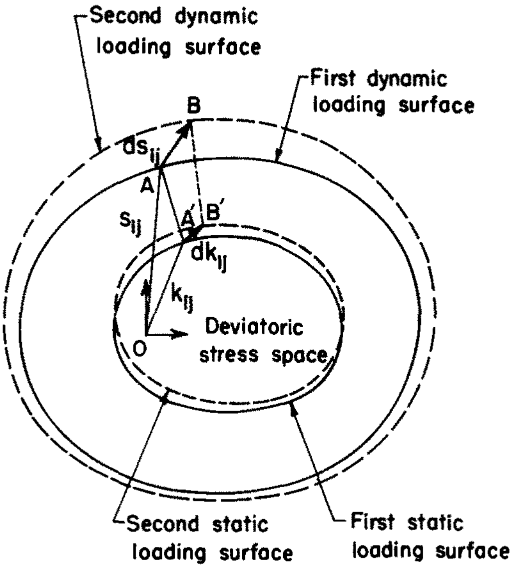

FIG. 4. The changes of yield and loading surfaces due to ds_{ij}.

When a ds_{ij} is applied at A, a plastic strain of $\dot{e}_{ij}\,dt = de_{ij}$ is produced. Since the vector $\dot{e}_{ij}$ is normal to both the first (subsequent) static yield surface and the first (subsequent) dynamic loading surface, the direction of the vector de_{ij} is known. The strain hardening theory of plasticity can now be used to find the second (subsequent) static yield surface.

Let the first static yield surface be denoted by

$$F(k_{ij}, e_{ij}, \kappa) = 0 \tag{21}$$

where κ is a strain-hardening parameter whose rate of change takes the special form

$$\dot{\kappa} = \dot{\kappa}(\dot{e}_{ij}) \tag{22}$$

and the rate of change of e_{ij} is specified by

$$\dot{e}_{ij} = g_{ij}(k_{rs}, \dot{k}_{rs}, e_{rs}). \tag{23}$$

After time dt has elapsed, plastic strain de_{ij} has been produced and the change of parameter κ has been $d\kappa$. Thus, the second static yield surface is

$$F(k_{ij}, e_{ij}+de_{ij}, \kappa+d\kappa) = 0. \tag{24}$$

The second dynamic loading surface is then simply a hypersurface passing through point B (with coordinates $s_{ij}+ds_{ij}$) and parallel to the second static yield surface. A perpendicular from B to the second static yield surface can be constructed, the intersecting point being point B'. Vector $A'B'$, which represents dk_{ij}, is thus obtained.

In obtaining the coordinates of point B', the method of Lagrangian multipliers can be used. The square of the distance between a point on surface (24) and the point B has to be minimized, i.e. we have to minimize the following expression:

$$g = [(s_{ij}+ds_{ij})-k_{ij}][(s_{ij}+ds_{ij})-k_{ij}]. \tag{25}$$

Introducing the Lagrangian multiplier λ, we obtain

$$\frac{\partial g}{\partial k_{ij}}+\lambda\frac{\partial F}{\partial k_{ij}} = 0. \tag{26}$$

Equations (26) and (24) constitute a set of nine equations with nine unknowns λ and k_{ij}.† Solving these nine equations simultaneously, we obtain the coordinates of point B'. The difference of the coordinates of the points A' and B' gives us the components of the vector dk_{ij}.

The changes of the yield surfaces due to the application of the static load dk_{ij} can be obtained by the same method discussed above. The application of dk_{ij} will produce the plastic strain de_{ij} which leads in turn to the determination of the second subsequent static yield surface. The second dynamic loading surfaces are then parallel to the second static yield surface.

All the discussions in this Section up to this stage refer to the case of "total loading" (to be defined later in this paper) where the infinitesimal loading ds_{ij} is pointing toward the exterior of the corresponding dynamic loading surface. For the case of "partial loading" (to be also defined later in this paper) where the infinitesimal loading ds_{ij} is pointing toward the interior of the corresponding dynamic loading surface, the above discussions remain valid. This is so, because all the conditions during the process of partial loading remain the same as those of the process of total loading, except that ds_{ij} is pointing inwards.

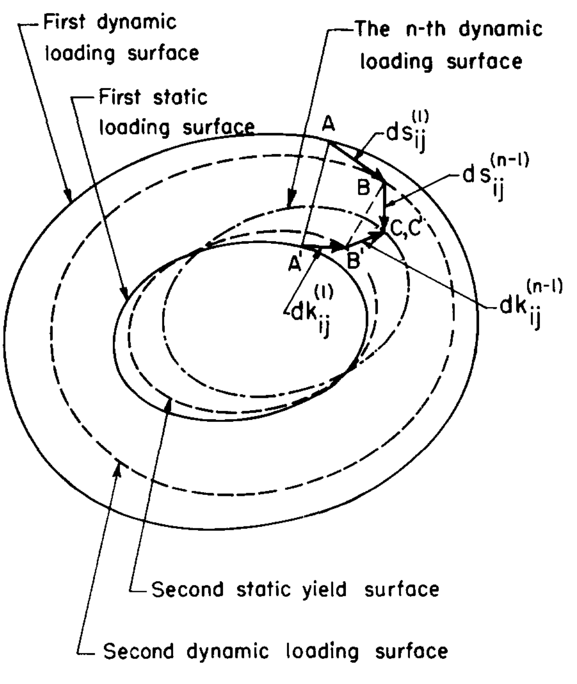

FIG. 5. The successive applications of ds_{ij} and the corresponding changes of the yield and loading surfaces during partial loading.

† Remember that k_{ij} are on the deviatoric plane, hence they follow the condition $k_{11}+k_{22}+k_{33} = 0$.

Figure 5 shows the successive applications of the infinitesimal loadings $ds_{ij}^{(1)}$, $ds_{ij}^{(2}$, ..., $ds_{ij}^{(n-1)}$ and the corresponding changes of the yield surfaces during the process of partial loading.

Let point A in Fig. 5 represent the present stress state and point A' be a corresponding point of A on the first static yield surface. If an inward $ds_{ij}^{(1)}$ is applied at A, we can obtain $dk_{ij}^{(1)}$ by use of the method discussed in the first part of the present section. Thus, the second static and dynamic surfaces can be obtained. They are shown by the dashed curves in Fig. 5. The same method applies, if, further, $ds_{ij}^{(2)}$, $ds_{ij}^{(3)}$, ..., etc. are applied successively. Thus, $dk_{ij}^{(2)}$, $dk_{ij}^{(3)}$, ..., etc. and also the 3rd, 4th, ... static and dynamic surfaces can be obtained. From Fig. 5 we see that as a result of successive applications of $ds_{ij}^{(1)}$, ..., etc. the dynamic loading surface "shrinks", while the static yield surface "expands" at the same time.

In the limit, after the $(n-1)$th infinitesimal load $ds_{ij}^{(n-1)}$ has been applied, the dynamic and the static surfaces will meet and coincide. This will happen, since the dynamic and static surfaces are parallel at all stages of partial loading. The distance h between the two surfaces is decreasing as the process of partial loading proceeds. In the limit, when h approaches to zero, the two surfaces coincide. The resulting surface is shown in Fig. 5 by a dash–dot–dash curve. Further application of infinitesimal loadings toward the interior of this surface will lead to "unloading".

4. NON-REGULAR QUASISTATIC YIELD SURFACES

It has been shown previously that the dynamic loading surface is parallel to the static yield surface with a distance of h between them and that this distance h increases with increasing strain rate. A method has been developed to obtain dk_{ij} for a given ds_{ij}. However, it was implicitly assumed that we have a regular static yield surface (i.e. a smooth surface without corner). In the present section, we shall extend the theory to include a static yield surface with corners. For simplicity, Prager's kinematic hardening rule will be used to find the changes of the static yield surface due to strain hardening.†

Let us now consider a static yield surface with a corner at A (see Fig. 6). To construct a dynamic loading surface based on this given static yield surface, let us remember that the dynamic loading surface is parallel to the static one with a distance h between them. Thus, surface BDC can be easily constructed, since the corresponding part of the static yield surface is regular. Surface BC is a part of a spherical surface having its center at A and its radius h. The construction of surface BC is justified, because it satisfies the following requirements:

(a) the distance between point A and any point on surface BC is h.

(b) for a regular dynamic loading surface, the strain rate vector $\dot{e}_{ij}$, lies along the direction of the outward drawn normal to the dynamic loading surface. The surface BC just constructed is a regular surface, therefore, normality holds on surface BC. Since BC is also a spherical surface, $\dot{e}_{ij}$ lies along the radial direction of the surface.

(c) Because $\dot{e}_{ij}$ lies along the radial direction, it lies always inside the cone BAC. This is consistent with the requirement of the theory of plasticity that $\dot{e}_{ij}$ should lie within the cone BAC if a stress increment is applied at corner A of the static yield surface.

† The kinematic hardening rule is used only for convenience. It is known from experiments, Phillips and Tang [13], that in reality a very different hardening rule is valid.

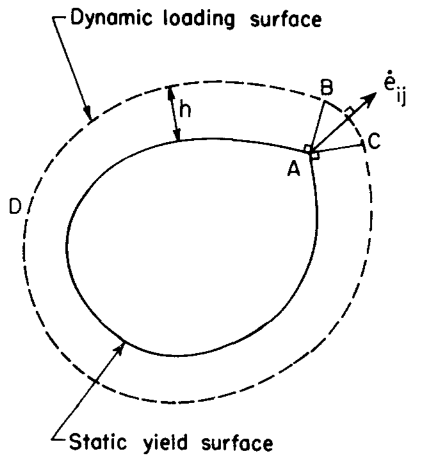

FIG. 6. A static yield surface with corner.

We further observe that corners on the static yield surface will not exist on the dynamic loading surfaces. With low strain rates, we have dynamic loading surfaces with rounded corners. No corner effects will be observed on the dynamic loading surfaces corresponding to higher strain rates.

Now, let us consider the problem which has been solved previously for the case of regular static yield surfaces; we have to find dk_{ij} for a given ds_{ij}. If the present stress point is located on the regular surface *BDC* (see Fig. 6), then the previous solution for a regular static yield surface remains valid. However, if the present stress point is on surface *BC*, then the corresponding vector dk_{ij} is located at the corner of the static yield surface. Three cases arise and will be considered separately as follows:

(a) The present stress point *A* lies on the intersection between the regular surface *AEG* and the spherical surface *AG*. The infinitesimal stress vector *AB* is pointing *toward* the spherical surface side (see Fig. 7).

The vector e_{ij} is normal to the first dynamic loading surface at *A* and makes an angle of 90° with the regular part of the first static yield surface. A plastic strain of $de_{ij} = \dot{e}_{ij}\,dt$ has been produced after time dt has elapsed. The appearance of this plastic strain de_{ij} enables us to determine the amount of translation of the static yield surface according to the kinematic hardening rule. Thus,

$$d\alpha_{ij} = c\,de_{ij} \tag{27}$$

where $d\alpha_{ij}$, is the translation of the static yield surface due to the application of vector ds_{ij}; c is a constant.

Since dk_{ij} is associated with point A' and a corner exists at point A', we have

$$dk_{ij} = d\alpha_{ij}. \tag{28}$$

Equation (28) is obtained, because we are considering the kinematic hardening rule where no rotations of the yield surface are allowed. Many possibilities of dk_{ij} exist which give rise to the same $d\alpha_{ij}$ (for example, dk_{ij} in direction $A'B''$, see Fig. 7, produces the same $d\alpha_{ij}$ as

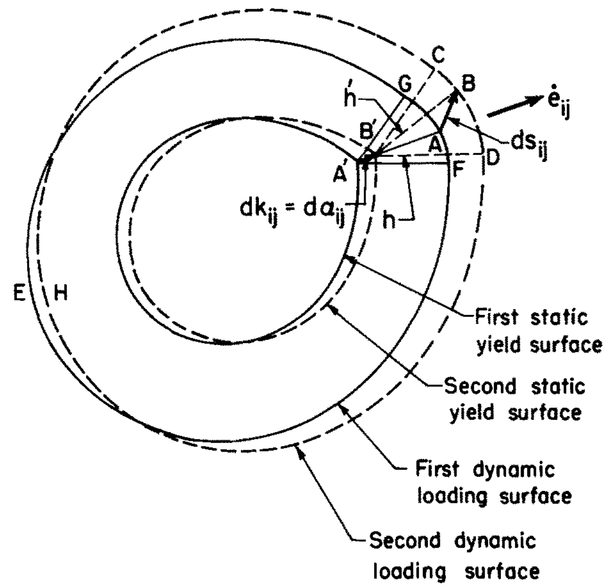

FIG. 7. The infinitesimal stress vector is pointing toward the spherical surface side.

dk_{ij} in direction $A'B'$ does). However, only the choice of expression (28) satisfies the requirement that the vector $(s_{ij}+ds_{ij})-(k_{ij}+dk_{ij})$ always has the magnitude h' (where h' is the excess stress corresponding to the specific strain rate after time dt has elapsed) and is always perpendicular to both the static and the dynamic surfaces.

The second dynamic loading surface can be constructed as follows: surface CHD is a surface which is parallel to the regular part of the second static yield surface with an excess stress h'. The scalar h' is equal to the length of the vector $B'B$. Surface CBD is part of the spherical surface with center at B' and radius h'.

(b) The present stress point A lies on the curve of intersection between the regular surface AEG and the spherical surface AG. The infinitesimal stress vector AB is pointing *away* from the spherical surface side (see Fig. 8).

The vector $\dot{e}_{ij}$ here is again normal to the first dynamic loading surface at A and makes an angle of 90° with the regular part of the first static yield surface. The plastic strain is again $de_{ij} = \dot{e}_{ij}\,dt$. By use of the kinematic hardening rule, $d\alpha_{ij}$ can be obtained which is represented in Fig. 8 by a vector $A'C'$. A perpendicular can be drawn from point B to the known second static yield surface. Thus, dk_{ij} can be found and is represented by the vector $A'B'$.

(c) The present stress point A lies on the spherical surface FAG (see Fig. 9).

In this case, the vector $\dot{e}_{ij}$ is normal to the first dynamic loading surface at A, and it lies within the cone $GA'F$ (i.e. within the limiting normals). According to the kinematic hardening rule at a corner, we then have

$$dk_{ij} = d\alpha_{ij}.$$

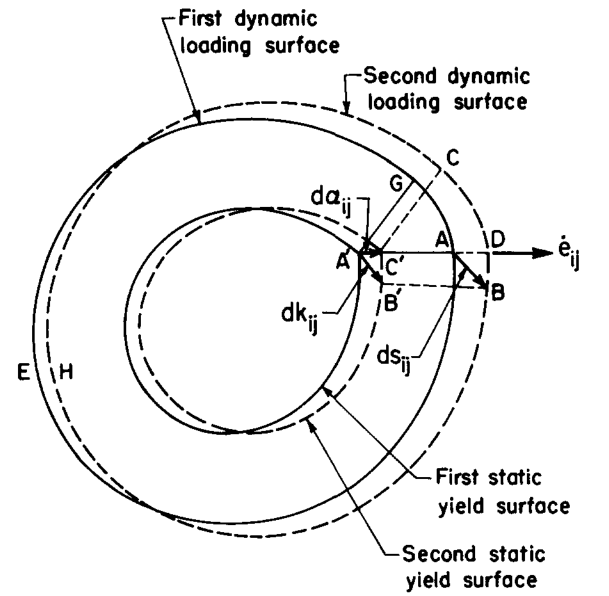

FIG. 8. The infinitesimal stress vector is pointing away from the spherical surface side.

This vector is represented by $A'B'$ in Fig. 9. The second dynamic loading surface can be constructed as follows: surface CHD is a surface parallel to the regular part of the second static yield surface with a distance h' between them, where h' is equal to the length of the vector $B'B$. Surface CBD is a part of the spherical surface with center at B' and radius h'.

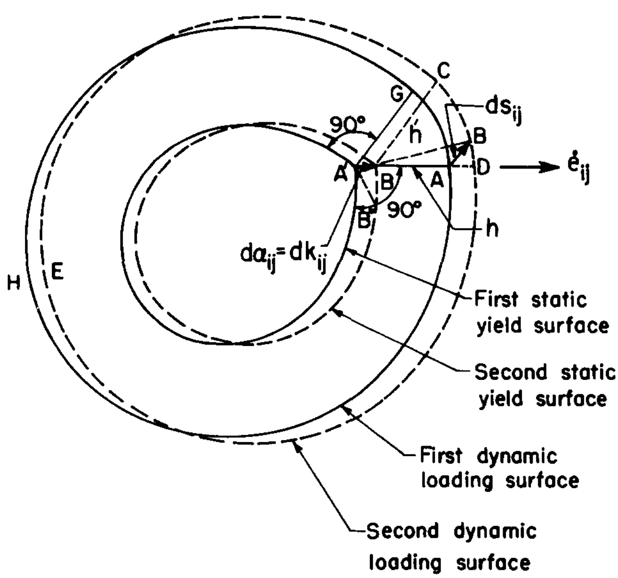

FIG. 9. The present stress point A lies on the spherical surface FAG.

5. LOADING/UNLOADING CRITERIA

In previous papers on viscoplasticity the rate of work $s_{ij}\dot{e}_{ij}$ has been used to define the loading/unloading criteria.† In the present theory, however, it is permitted that the origin may lie outside the dynamic loading surface. In such a case the previous definitions are not suitable. Indeed let us consider the case illustrated by Fig. 10. Let the present state of stress be at A. If an additional loading ds_{ij} is applied at A pointing toward the exterior of the dynamic loading surface then the plastic strain rate vector $\dot{e}_{ij}$, which is normal to the dynamic loading surface at A, makes an obtuse angle with the vector s_{ij}. The rate of work $s_{ij}\dot{e}_{ij}$ is negative, and according to previous terminology f.e. Cristescu [14] it corresponds to "quasi-unloading" while in reality it corresponds to "loading."

We shall therefore introduce new loading/unloading criteria based on the excess stress vector h_{ij}. The dashed lines and the numbering in Fig. 11 correspond to the loading/unloading cases as follows:

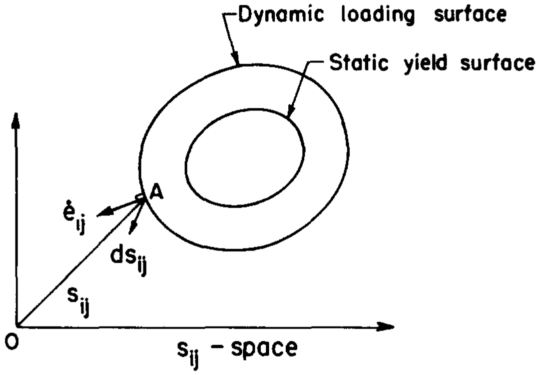

FIG. 10. The use of $s_{ij}\dot{e}_{ij}$ limits the applicability of the loading/unloading criterion.

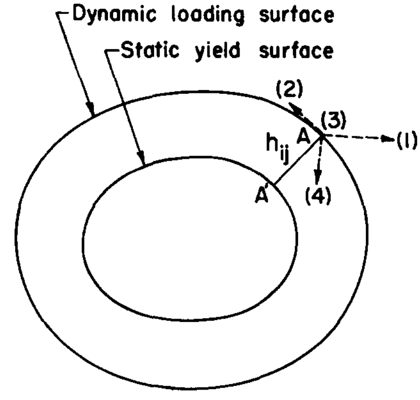

FIG. 11. The new loading/unloading criterion.

† Cristescu [14], also see Perzyna [15].

1. *Total loading*

The material is said to experience a "total loading" process, if the following condition is satisfied

$$h_{ij}\,\mathrm{d}s_{ij} > 0. \tag{29}$$

During this process, additional plastic strain occurs, i.e. $\dot{e}_{ij} > 0$. Both the static and the dynamic yield surfaces "expand." By "expanding" we mean that the yield surface moves with the loading point toward the exterior of the initial yield surface and changes its shape when the material experiences strain-hardening. In particular, if the material obeys the isotropic strain-hardening rule, the word "expand" has its real meaning. This process is marked by (1) in Fig. 11.

2. *Neutral loading*

The material is said to experience a "neutral loading" process, if the following condition is satisfied:

$$h_{ij}\,\mathrm{d}s_{ij} = 0. \tag{30}$$

During this process, additional plastic strain also occurs, i.e. $\dot{e}_{ij} > 0$. The stress point A moves on the dynamic loading surface at some speed, but may not stop at any one point on the surface. Otherwise, creep will occur. The dynamic loading surface remains unchanged, but the static yield surface "expands". This process is marked by (2) in Fig. 11.

3. *Creep*

The material is said to creep, if the following condition is satisfied:

$$\mathrm{d}s_{ij} = 0 \tag{31}$$

i.e. the stress point is stationary on the dynamic loading surface. During this process, additional plastic strain occurs, i.e. $\dot{e}_{ij} > 0$. The dynamic loading surface remains unchanged, while the static yield surface "expands". The creep rate decreases during the process because the excess stress h decreases with time. This point can be explained best by considering the one-dimensional case.

In such a case a line parallel to the strain axis represents the creep process. It is readily seen that the excess stress h above the static curve decreases with the increasing plastic strain. Since in viscoplasticity the plastic strain takes time to occur h decreases with time. According to the present theory, the creep rate is a function of h and it decreases when h decreases. Therefore the creep rate decreases with time. The creep process is marked by (3) in Fig. 11.

4. *Partial loading*

The material is said to experience a "partial loading" process, if the following conditions are satisfied:

$$h_{ij}\,\mathrm{d}s_{ij} < 0 \quad \text{and} \quad \dot{e}_{ij} > 0. \tag{32}$$

During this process, additional plastic strain occurs. The dynamic loading surface "shrinks", while the static yield surface "expands". By "shrinking" we mean that the dynamic loading surface moves with the loading point toward the interior of the initial dynamic loading surface and changes its shape when the material strain-hardens. This process is marked by (4) in Fig. 11.

5. *Unloading*

Two cases are to be considered.

(a) Instantaneous decrease of stress is allowed: in this case, the following conditions have to be satisfied:

$$h_{ij}\,\mathrm{d}s_{ij} < 0 \quad \text{and} \quad \dot{e}_{ij} = 0. \tag{33}$$

During this process, no additional plastic strain occurs. The material is rigid and the dynamic loading surface "shrinks" to become coincident with the static yield surface, while the latter remains unchanged.

(b) No instantaneous decrease of stress is allowed: unloading takes place when the "shrinking" dynamic yield surface and the "expanding" static yield surface meet and coincide. The material is rigid during this process.

It is to be remarked here, that no relaxation occurs, since the elastic strain has been neglected.

6. CONCLUSIONS

In this paper we presented an alternative to the currently used Perzyna's theory of viscoplasticity.

An examination of the two theories shows that they are equivalent only when the quasistatic yield surface is the Mises yield surface. When the quasistatic yield surface is different than the J_2-one, the new theory allows for a quasistatic yield surface and a dynamic loading surface which conceptually are simpler than the ones introduced by Perzyna.

In our theory the plastic strain rate vector is normal to the quasistatic yield surface while in Perzyna's theory it is not. In addition new loading/unloading criteria has been introduced.

Experimental work to show which one of the two theories gives results in better agreement with reality is necessary. A study of presently available experimental evidence and its interpretation concerning the new as well as Perzyna's theory will be given in a subsequent paper.

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Абстракт—Дается вариант вообще использованной теории Пежины касающейся вязкопластичности для случая малых деформаций. Вектор скорости пластической деформации является нормальным к квазистатической поверхности течения, которая при обороте может не заключивать начала. Последовательно, определяются новые критерия для нагрузки и разгрузки.